Determination of the Limit Cycle by He's Parameter-Expansion for Oscillators in a $u^3/(1+u^2)$ Potential

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This paper applies He's parameter-expansion method to determine the limit cycle of oscillators in a $u^3/(1+u^2)$ potential. The results are compared with the exact solutions. This shows that the method is a convenient and powerful mathematical tool for the search of limit cycles of nonlinear oscillators.

Key words: He's Parameter-Expansion Method (PEM); Limit Cycle; Nonlinear Oscillators.

1. Introduction

Recently some new methods for nonlinear equations have caught broad attention, for example the variational iteration method [1 – 5], the variational approach [6 – 9], the homotopy perturbation method [10 – 21] and the parameter-expansion method [22 – 26]. In the parameter-expansion method a constant, rather than a nonlinear frequency is expanded in powers of the expanding parameter to avoid the occurrence of a secular term in the perturbation series solution. It is proven that the parameter-expansion methods are very effective to determine the limit cycle of strongly nonlinear oscillators with high accuracy. This paper applies the method to determine the limit cycle of oscillators in a $u^3/(1+u^2)$ potential.

2. He's Parameter-Expansion Method [23]

We consider the following nonlinear oscillator

$$u'' + \frac{u^3}{1+u^2} = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{1}$$

Equation (1) can be rewritten in the form

$$\begin{aligned} &u'' + 0\cdot u + 1\cdot\frac{u^3}{1+u^2} = 0,\\ &u(0) = A, \quad u'(0) = 0. \end{aligned} \tag{2}$$

According to parameter-expansion methods [22 – 26], we assume that the solution is expanded into a series of p in the following form:

$$u = u_0 + pu_1 + p^2u_2 + \cdots, \tag{3}$$

where p is a book-keeping parameter, $p = 1$.

The coefficients 0 and 1 can be, respectively, expanded into a series in p in a similar way [11]:

$$0 = \varpi^2 + p\varpi_1 + p^2\varpi_2 + \cdots, \tag{4}$$

$$1 = pc_1 + p^2c_2 + \cdots. \tag{5}$$

Substituting (3), (4) and (5) into (2) and processing with the standard perturbation method, we have

$$p^0: u_0'' + \varpi^2u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0, \tag{6}$$

$$p^1: u_1'' + \varpi^2u_1 + \varpi_1u_0 + c_1\frac{u_0^3}{1+u_0^2} = 0. \tag{7}$$

Solving (6), we have

$$u_0 = A\cos\varpi t. \tag{8}$$

Substituting (8) into (7) results in

$$u_1'' + \varpi^2u_1 + \varpi_1A\cos\varpi t + c_1\frac{A^3\cos^3\varpi t}{1+A^2\cos^2\varpi t} = 0. \tag{9}$$

We expand the term $\frac{A^3\cos^3\varpi t}{1+A^2\cos\varpi t}$ into a Fourier series representation as follows:

$$\begin{aligned} \frac{A^3\cos^3\varpi t}{1+A^2\cos^2\varpi t} &= \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\varpi t\\ &= a_1\cos\varpi t + a_3\cos 3\varpi t + \cdots, \end{aligned} \tag{10}$$

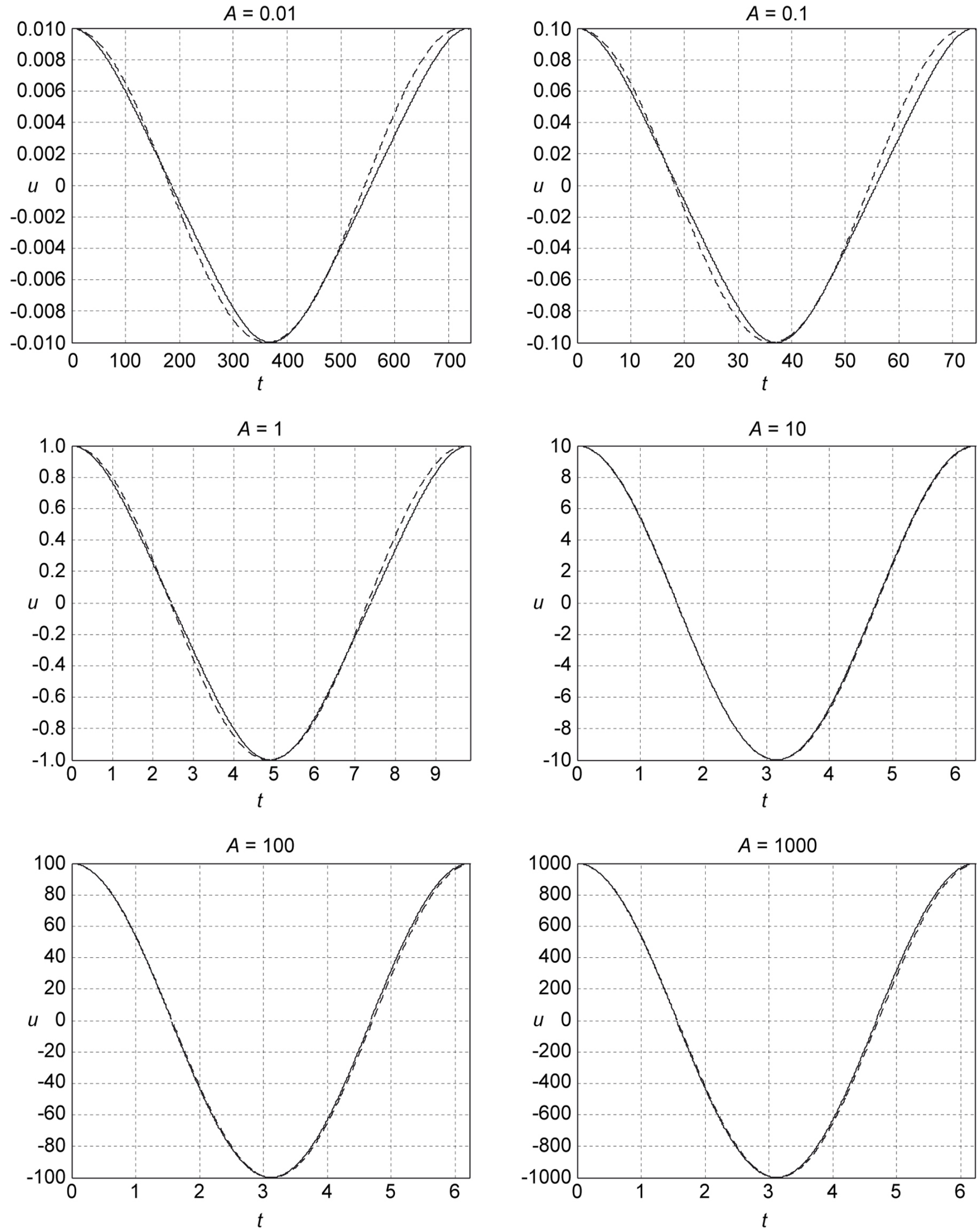

Fig. 1. Comparison of the approximate solution with the exact solution; dashed line, approximated solution; solid line, exact solution.

with

$$a_1 = \frac{4}{\pi}\int_0^{\pi/2} \frac{A^3\cos^3\varpi t}{1+A^2\cos^2\varpi t}\varpi\cos\varpi t\,\mathrm{d}t = A+\frac{2}{A}\left(\frac{2}{\sqrt{1+A^2}}-1\right),$$

and the interval of t in (10) is $[\frac{-\pi}{\omega}\,\frac{\pi}{\omega}]$. Therefore, the first several terms are

$$\frac{A^3\cos^3\varpi t}{1+A^2\cos^2\varpi t} = \left[A+\frac{2}{A}\left(\frac{2}{\sqrt{1+A^2}}-1\right)\right] \cdot\cos\varpi t+\sum_{n=1}^{\infty} a_{2n+1}\cos(2n+1)\omega t. \tag{11}$$

Substituting (11) into (9) and $c_1 = 1$ (for $p = 1$) yields

$$u_1'' + \varpi^2 u_1 + \varpi_1 A \cos \varpi t + c_1 \left\{ \left[A + \frac{2}{A} \left(\frac{2}{\sqrt{1+A^2}} - 1 \right) \right] \cos \varpi t + \sum_{n=1}^{\infty} a_{2n+1} \cos(2n+1)\omega t \right\} = 0. \quad (12)$$

The requirement of no secular term gives

$$\varpi_1 A + A + \frac{2}{A} \left(\frac{1}{\sqrt{1+A^2}} - 1 \right) = 0, \quad (13)$$

then

$$\varpi_1 = \frac{2}{A^2} - \frac{2}{A^2\sqrt{1+A^2}} - 1. \quad (14)$$

If the first-order approximation is enough, then, setting $p = 1$ in (4) and (5), we have

$$\varpi^2 + \varpi_1 = 0, \quad (15)$$

$$c_1 = 1. \quad (16)$$

Solving (13) – (16), the frequency can be obtained in the form

$$\varpi = -\sqrt{\frac{2}{A^2\sqrt{1+A^2}} + 1 - \frac{2}{A^2}}.$$

Hence, we can obtain the following zero-order approximate solution:

$$u = A \cos \left(-\sqrt{\frac{2}{A^2\sqrt{1+A^2}} + 1 - \frac{2}{A^2}t} \right), \quad (17)$$

which agrees very well with the exact solution, as shown in Figure 1.

3. Conclusion

In this paper, we apply He's parameter-expansion method to obtain the highly accurate approximate periodic solutions of nonlinear oscillators in an $u^3/(1+u^2)$ potential. The results obtained comparing with the exact solutions show that the method is a convenient and powerful mathematical tool for the search of limit cycles of nonlinear oscillators and can be easily extended to any nonlinear problem.

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